Introduction

Def: A *differential equation* is an equation involving one or more derivatives or differentials of an unknown function.

Ex:

1) $\frac{dy}{dx} = 3x + 1$

2) $e^y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$

3) $\frac{d^3x}{dt^2} + (\cos t)\frac{d^2x}{dt^2} + xt = 0$

4) $(f''(x))^3 + 3(f'(x))^4 + x^3(f'(x))^2 = x$

5) $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

are all differential equations.

Def: If a differential equation involves only ordinary derivatives of a function of one variable it is an *ordinary differential equation* (ODE). If it involves partial derivatives of a function of two or more variables it is a *partial differential equation* (PDE). This class is concerned with ODEs!

Ex: 1–4 above are ODES. 5 is a PDE.

Def: The *order* of a DE is the order of the highest order derivative which appears. The *degree* of a DE is the power to which the highest order derivative appears.

Ex: 1 above is first order, first degree
2 is second order, first degree
3 is third order, first degree
4 is second order, third degree
5 is second order, first degree

Almost all equations you will encounter as an undergrad are first or second order of degree one.

Note w.r.t. $\equiv$ "with respect to"

Notation –

If $y = y(x) = f(x)$ then $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{(n)}y}{dx^{(n)}}$ or $y', y'', \ldots, y^{(n)}$ represent the first, second, ..., n^{th} derivatives of y w.r.t. x.

So $y^{(3)}(x) = y'''(x) = \frac{d^3y}{dx^3} = 3^{rd}$ derivative of y w.r.t. x.

The independent variable is frequently time, t. Physicists and engineers use dots to denote derivatives w.r.t. time.

So if $x = x(t)$ then $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, etc....

If $z = z(t)$ then $\dot{z} = \frac{dz}{dt}$, $\ddot{z} = \frac{d^2z}{dt^2}$, etc....

* Note: you will see notation like $F(x, y, y', \ldots, y^{(n)}) = 0$ I rarely use it.

Solutions – a function $y = y(x)$ is a solution of a DE if it satisfies the equation.

Ex: $y = c_1 \sin(2x) + c_2 \cos(2x)$ is a solution of the ODE for c_1, c_2 constants

$y'' + 4y = 0$ because

$$y' = 2c_1 \cos(2x) - 2c_2 \sin(2x)$$

$$y'' = -4c_1 \sin(2x) - 4c_2 \cos(2x)$$

So $y'' + 4y = -4c_1 \sin(2x) - 4c_2 \cos(2x) + 4(c_1 \sin(2x) + c_2 \cos(2x)) = 0$

A DE may have zero solutions, a finite number of solutions, or an infinite number of solutions.

The set of all solutions is called the <u>general solution</u>.

Any one solution is a <u>particular solution</u>.

So $y = c_1 \sin(2x) + c_2 \cos(2x)$ is the general solution of $y'' + 4y = 0$. i.e. it is an infinite set of solutions, one for each choice of c_1 and c_2.

But
$$y = 3\sin(2x) + 12\cos(2x)$$
$$y = -\sin(2x) + \pi \cos(2x)$$
$$y = \frac{1}{7}\sin(2x) - 137\cos(2x)$$
are each particular solutions.

As a rule, the general solution of an n^{th} order ODE will contain n constants. We'll come back to this but it's easy to see that solution of an n^{th} order equation requires n integrations, each introducing a constant.

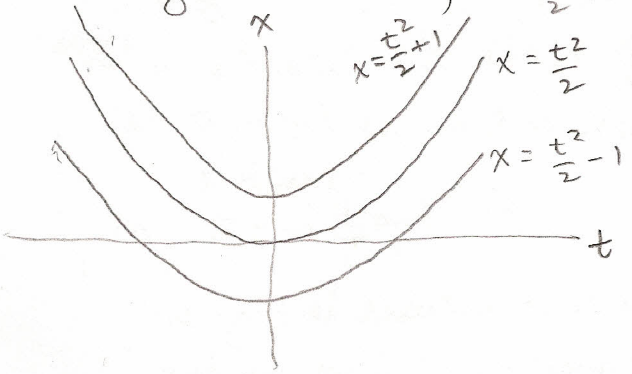
As an example, consider the first order equation $\dot{x} = t$ or $\frac{dx}{dt} = t$.

If $\frac{dx}{dt} = t$, then $x = \int t\,dt$

$$x = \frac{t^2}{2} + c, \quad c = \text{constant}$$

You can show that $x = \frac{t^2}{2} + c$ satisfies $\frac{dx}{dt} = t$ for any constant c.

This general solution, $x = \frac{t^2}{2} + c$, is actually an infinite set of solutions.



You can't draw them all but the infinite set of solutions fills the $t-x$ plane.

Now consider the 2^{nd} order equation

$$\frac{d^2x}{dt^2} = t$$

then $$\frac{dx}{dt} = \frac{t^2}{2} + c_1$$

$$x = \frac{t^3}{6} + c_1 t + c_2 \quad \text{(you show this is a solution)}$$

and the general solution contains two constants, c_1 and c_2.

In general, an n^{th} order ODE requires n integrations, each introducing another constant.

Initial Value problems and Boundary value problems –

Since the general solution of an n^{th} order equation contains n unknown constants we require n conditions to find any particular solution (of course, a given physical problem has one solution).

If the n conditions are all given at the same value of the independent variable then they are initial conditions and the problem is an initial value problem (IVP). If the conditions are given at different values they are boundary conditions and the problem is a boundary value problem (BVP).

Historically, this terminology comes from physical problems.
In dynamics, position, velocity, etc. are functions of time and if we give enough conditions at time $t=0$ (the initial time) we can find a particular solution for any later time.

For example, for an object moving in 1-dimension where position is given by $x = x(t)$, velocity $v = v(t) = \frac{dx}{dt} = \dot{x}$ and acceleration $= a = a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x}$.

So Newton's second law becomes -

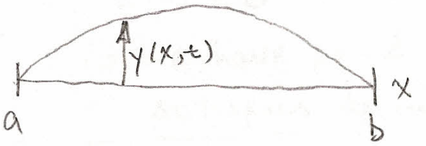
$$F = ma$$

or $$F = m\frac{d^2x}{dt^2}$$

The general solution to this 2nd order ODE contains 2 constants and we need 2 conditions to solve for those constants to find the particular solution corresponding to the physical situation at hand.

If we give position and velocity at time $t=0$ we have the initial value problem $F = m\frac{d^2x}{dt^2}$, $x(0) = x_0$, $v(0) = v_0$ where x_0 and v_0 are the position and velocity we measure at time $t=0$.

Now suppose we want to solve for vibrations of a string fixed at 2 ends, that is, the displacement y of the string for any time t and position x.



Don't worry about solution, this is a 2nd order PDE, but the general solution contains 2 constants so we need 2 conditions.

If we know the string is fixed at $x=a$ and $x=b$ then we have a 2nd order equation plus the 2 conditions $y(a) = y(b) = 0$ for all time t. This is a BVP because the conditions are given at the two boundaries.

<u>Example</u> - Solve the IVP

$$\ddot{x} = e^t, \quad x(0) = 0, \quad \dot{x}(0) = 1$$

We can do this integration –

$$\frac{d^2x}{dt^2} = e^t$$

$$\frac{dx}{dt} = \int e^t dt$$

$$= e^t + c_1$$

$$x = \int (e^t + c_1) dt$$

$$= e^t + c_1 t + c_2$$

Now we apply the conditions –

$$x(0) = e^0 + c_2 = 0$$

so $1 + c_2 = 0$.

$$\dot{x}(0) = e^0 + c_1 = 0$$

$$1 + c_1 = 0$$

So we find $c_1 = c_2 = -1$

and the solution to the IVP is

$$x(t) = e^t - t - 1$$

this can be checked –

$\ddot{x} = \frac{d^2x}{dt^2} = e^t$ and the solution satisfies the ODE.

It must also satisfy the two conditions –

$$x(0) = e^0 - 0 - 1 = 0$$

$$\dot{x}(0) = e^0 - 1 = 0$$

So we have found the particular solution to the ODE which satisfies the initial conditions and thus we have solved the IVP.

The solution of a BVP is similar –

Suppose we have the BVP $\ddot{x} = e^t$, $x(0) = 0$, $x(1) = 2$.

We know $x = e^t + c_1 t + c_2$ is the general solution. Now we satisfy the boundary conditions –

$$x(0) = e^0 + c_2 = 1 + c_2 = 0$$

so $c_2 = -1$

$$x(1) = e^1 + c_1 + c_2$$

$$= e + c_1 - 1$$

$$= 2$$

so $c_1 = 3 - e$

and $x = e^t + (3-e)t - 1$ is the solution of the BVP. You verify!
